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Modeling of the fluctuating component in the form of the sum of an infinite number of random quantities. Part 1: k–ε Modeling

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ABSTRACT

A new approach to construction of turbulence models that allows modeling of fluctuating components in the form of an infinite number of random quantities has been suggested. The first part of the work deals with the technique of construction of the k–ε-type model. The calculations show a great expansion of the possibilities of models. Among the problems under solution there are calculations of spectrum, coherent structures, modeling of the bypass transition, etc. The theory that forms the basis of the approach is confirmed experimentally.

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1. Introduction

At present, the approach based on the time-averaged Navier–Stokes equations is the most economically substantiated approach to turbulence modeling. In this approach, flow velocity, temperature, pressure, etc. are divided to the sum of the averaged and random components, with the random component being determined as the difference between the instantaneous and averaged values of the variable. In the overwhelming majority of modern models, the random component is described by a single random quantity with zero mathematical expectation. To obtain model equations, the instantaneous velocity, pressure, temperature, etc. in the Navier–Stokes equations are substituted by the sum of the averaged and random quantities. The result of this substitution is averaged according to the expression $\overline{A(x,\tau)} = \lim_{T\to\infty}\frac{1}{2T}\int_{t-T}^{t+T}A(x,\tau)dt$ (the so-called Reynolds averaging). All k–ε, k–ω, k–L, etc. models, RNG models, models of transfer of correlations of the second and/or third order, etc. belong to the models constructed on the basis of this approach.

It is known from the experimental data that turbulence is a set of vortices which is distinguished by a great variety of sizes, time scales, etc. It seems evident that as a result of presentation of the fluctuating component in the form of a single random quantity and subsequent averaging many of the important properties of the random field are irretrievably lost.

The limitedness of the approach was noticed by Townsend [1] in his theoretical investigation of the turbulent boundary layer. To overcome the deadlock condition he suggested to present the instantaneous velocity as a sum of three terms: averaged velocity + long-wave fluctuations + medium-wave fluctuations. In other words, Townsend [1] suggested a description of the random component in the form of the sum of two random quantities. To calculate long-wave and medium-wave parts of the fluctuation spectrum separately was also suggested in the works of Kovasznay [2], Pedisius and Shlanciauskas [3], and Reynolds and Hussain [4]. We note that no model systems of equations were suggested in these works.

In the present paper, we suggest a technique for construction of turbulence models that are based on time-averaging, which allows presentation of the fluctuating component in the form of the sum of an infinite number of random quantities.

2. Equation of turbulence energy transfer for the case of presentation of fluctuating components as a sum of random quantities

The technique of obtaining the equation of turbulence energy transfer for the case of presenting the turbulence in the form of a single quantity is well known. A thorough description of this procedure can be found in the work of Hinze [5]. The same work also states that an exact transport equation can be written in the form

$$\frac{\partial}{\partial x_i}\overline{U}_i k = \nu\frac{\partial^2 k}{\partial x_k \partial x_k} + \mathrm{Diff}_{\mathrm{turb}}(k) + P - \varepsilon. \quad (1)$$

Here, $k \equiv 0.5\overline{u_i u_i}$ is the kinetic energy of turbulence. Here and below summation over repetitive indexes is supposed.

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Nomenclature

C_1, C_2, C_ε, C_v constants of the turbulence model
C_f friction coefficient
E spectral function
E_i dimensionless discrete spectral function, $1/\kappa k_i/\sum k_i 1$
F_v, f_0, $f_{0\text{-}1}$, $f_{0\text{-}i}$ functions of the turbulence model
H form parameter
k total energy of turbulence
k_0, k_1, k_i components of the turbulence energy
L_ε dissipative scale, $k^{3/2}/\varepsilon$
Nu Nusselt number
P generation of turbulence energy, $v_t(\partial U/\partial y)^2$
Re, Re_x Reynolds number, $U_e x/v$
Re_y turbulent Reynolds number, $\sqrt{k}y/v$
$\mathrm{Tu_e}$ level of outer turbulence, $\sqrt{2/3k_e}/U_e$
ΔT temperature difference, $T_w - T_e$
t' temperature fluctuation
U, $\overline{U}$ instantaneous and mean velocities in the x direction
U', u'_0, u'_1, u'_i fluctuation components of velocity in the x direction

Greek symbols
δ boundary-layer thickness
ε dissipation rate k, $\overline{v(\partial u_i/\partial x_i)^2}$
κ wave number

Subscripts
e in a free flow
w on a wall

The terms of this equation are interpreted as follows: $\mathrm{Diff_{turb}}(k) \equiv -\frac{\partial}{\partial x_j}\overline{u_j\left(\frac{p}{\rho}+\frac{u_i u_i}{2}\right)}$ is the turbulent diffusion, $P \equiv -\overline{u_i u_j}\frac{\partial \overline{U_j}}{\partial x_i}$ is the energy generation due to average flow, and $\varepsilon \equiv 2v\overline{\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$ is the rate of turbulence energy dissipation. The generation term is given in Eq. (1) exactly. Turbulent diffusion and dissipation rate must be modeled.

Expression (2) is a standard model for turbulent diffusion

$$\mathrm{Diff_{turb}}(k) = \frac{\partial}{\partial x_k}\frac{v_t}{C_k}\frac{\partial k}{\partial x_k}. \tag{2}$$

Here, v_t is the eddy viscosity that, as a rule, is calculated by the Kolmogorov–Boussinesq relation; C_k is the model constant.

By analogy with this technique, we present the instantaneous values of the velocity and pressure in the form of the sum of one averaged and two fluctuating components $U_j = \overline{U_j} + u_{0j} + u_{1j}$, $P = \overline{P} + p_0 + p_1$ and substitute this presentation into the Navier–Stokes equation. Generally speaking, now it is necessary to perform averaging. But in the case of presentation of the fluctuating component as a sum of several random quantities the averaging procedure requires additional comments.

According to the definition, the Reynolds averaging is as follows – $\overline{A} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\,dt$. The fluctuating component is found as a difference between the instantaneous and average values. But this procedure can distinguish only the average and fluctuating components of the instantaneous velocity. In fact, according to this definition we obtain that $\overline{(U+\sum u_i)(V+\sum v_i)} = \overline{(U+u)(V+v)}$. Here, $u = \sum u_i$, $v = \sum v_i$. In other words, the result of averaging does not depend on the way of presentation of the fluctuating component.

To avoid such situation, the random component must be subject to filtering. Actually this means that if we wish to present the random component as a sum of random quantities $u = \sum u_i$, we need to have a set of filtration functions of the following form:

$$\varphi_i(u) = \begin{cases} 1 & u = u_i, \\ 0 & u \neq u_i. \end{cases}$$

Then, presenting the random component as $u = u\sum_i \phi_i(u)$ and performing multiply the above averaging, we can obtain the desired result. The complete procedure of averaging and filtering is as follows. Let us need to find $\overline{(U+u_1+u_2)(U+u_1+u_2)}$. At the first step we find the average velocity $\overline{U}$ and subtracting it from the instantaneous velocity we find the random component $u = u_1 + u_2$. Then, using standard averaging, we find $\overline{\varphi_1^2(u)uu} = \overline{u_1^2}$. Then we find $\overline{\varphi_1(u)u\varphi_2(u)u} = \overline{u_1 u_2}$ and so on. It should be noted that development of such a system of functions is a complex problem that, most probably, can be solved only in a trivial case, e.g., in separation of fluctuations by frequency. Here, we do not construct any specific system of filtering functions but only presuppose its existence. The more so as the meaning of random terms has not been determined as yet.

As a result we obtain an exact equation of the transfer of turbulence kinetic energy for the case of presentation of turbulence in the form of the sum of two random quantities

$$\begin{aligned}
&\frac{\partial}{\partial x_i}\overline{U_i}k_0 + \frac{\partial}{\partial x_i}\overline{U_i}k_1 \\
&\quad = v\frac{\partial^2 k_0}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_j}\overline{u_{0j}\left(\frac{p_0}{\rho}+\frac{u_{0i}u_{0i}}{2}\right)} - v\overline{\frac{\partial u_{0j}}{\partial x_k}\frac{\partial u_{0j}}{\partial x_k}} \\
&\qquad + v\frac{\partial^2 k_1}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_j}\overline{u_{1j}\left(\frac{p_1}{\rho}+\frac{u_{1i}u_{1i}}{2}\right)} - v\overline{\frac{\partial u_{1j}}{\partial x_k}\frac{\partial u_{1j}}{\partial x_k}} \\
&\qquad - (\overline{u_{0i}u_{0j}} + \overline{u_{1i}u_{0j}} + \overline{u_{0i}u_{1j}} + \overline{u_{1i}u_{1j}})\frac{\partial \overline{U_j}}{\partial x_i} \\
&\qquad + \frac{\partial}{\partial x_i}\overline{U_i}\overline{u_{0j}u_{1j}} + v\frac{\partial^2}{\partial x_k \partial x_k}\overline{u_{0j}u_{1j}} - \frac{1}{\rho}\overline{u_{1j}\frac{\partial p_0}{\partial x_j}} - \frac{1}{\rho}\overline{u_{0j}\frac{\partial p_1}{\partial x_j}} \\
&\qquad - 2v\overline{\frac{\partial u_{1j}}{\partial x_k}\frac{\partial u_{0j}}{\partial x_k}} - \frac{1}{2}\frac{\partial}{\partial x_i}\overline{u_{0i}u_{1j}u_{1j} + 2u_{0i}u_{1j}u_{0j} + u_{1i}u_{1j}u_{0j}}.
\end{aligned} \tag{3}$$

Here, we used the notation: $k_0 = 0.5\overline{u_{0i}u_{0i}}$ and $k_1 = 0.5\overline{u_{1i}u_{1i}}$.

The terms of this equation, according to their position in the lines, are interpreted as follows: 1st line – convective transfer of k_0 and k_1; 2nd line – diffusion and dissipation of k_0; 3rd line – diffusion and dissipation of k_1; 4th line – total energy transferred from the averaged flow to turbulence; 5th and 6th lines – transfer of the energy of interaction of vortex systems.

The most important hypothesis underlying further simplification of (3) is in the assumption that the energy of interaction of fluctuations can be neglected. Physical grounds of such an assumption can be very different. For example, fluctuations act in nonintersecting parts of the spectrum (this hypothesis was formulated by Townsend [1]).

From the viewpoint of construction of the model the hypothesis on the smallness of interaction leads to the following. First, we discard the terms u_{0i} and u_{1i} in the 4th line of Eq. (3). Second, by the same reason we discard all terms in the 5th and 6th lines of Eq. (3). As a result we have

$$\begin{aligned}\frac{\partial}{\partial x_i}\overline{U_i}k_0 + \frac{\partial}{\partial x_i}\overline{U_i}k_1 &= v\frac{\partial^2 k_0}{\partial x_k \partial x_k} + \mathrm{Diff_{turb}}(k_0) + v\frac{\partial^2 k_1}{\partial x_k \partial x_k} \\ &\quad + \mathrm{Diff_{turb}}(k_1) + P - \varepsilon\end{aligned} \tag{4}$$

Here, we used the notation

$$\mathrm{Diff}_{\mathrm{turb}}(k_0) = -\frac{\partial}{\partial x_j}\overline{u_{0j}\left(\frac{p_0}{\rho} + \frac{u_{0i}u_{0i}}{2}\right)},$$
$$\mathrm{Diff}_{\mathrm{turb}}(k_1) = -\frac{\partial}{\partial x_j}\overline{u_{1j}\left(\frac{p_1}{\rho} + \frac{u_{1i}u_{1i}}{2}\right)},$$
$$P_0 = -\overline{u_{0i}u_{0j}}\frac{\partial \overline{U_i}}{\partial x_j}, \quad P_1 = -\overline{u_{1i}u_{1j}}\frac{\partial \overline{U_i}}{\partial x_j}, \quad P = P_0 + P_1,$$
$$\varepsilon_0 = \nu\overline{\frac{\partial u_{0j}}{\partial x_k}\frac{\partial u_{0j}}{\partial x_k}}, \quad \varepsilon_1 = \nu\overline{\frac{\partial u_{1j}}{\partial x_k}\frac{\partial u_{1j}}{\partial x_k}}, \quad \varepsilon = \varepsilon_0 + \varepsilon_1.$$

Further simplification is in splitting of (4) to two separate equations. But, if convective and diffusion terms are distributed in the equations quite obviously, then it is impossible to distribute the generation and dissipative terms by two equations not resorting to additional hypothesis.

An example of such splitting is a multiscale model of Hanjalic et al. [6]. The authors of Ref. [6] divided the vortices participating in the cascade process to two parts, with the total energy of turbulence being taken equal to the sum of energies of the parts, $k = k_0 + k_1$. It was assumed that vortices with the energy k_0 interact with the middle flow, i.e., they take energy from this flow, whereas vortices with the energy k_1 transfer this energy to heat. In other words, vortices having the energy k_0 produce long-wave fluctuations and vortices having the energy k_1 produce middle-wave plus short-wave fluctuations. This assumption allows one to neglect the energy of interaction since the vortices act in different pars of the spectrum.

In this work, we suggest to make splitting based on quite different physical grounds.

3. Some properties of laminar vortex flows

It is natural that laminar and turbulent flows differ noticeably by their physical characteristics. At the same time, thorough consideration of the behavior of laminar vortices allows many important conclusions on the nature of turbulence. The theory of extension of vortex tubes, on whose basis the most important conclusions on the essence of the cascade process were drawn, can serve a classical example of such transfer of the properties of laminar flows on turbulent ones.

The fact of the appearance of secondary, tertiary, etc. vortices due to the contact between the vortex and the wall is well known in vortical near-wall laminar flows. Examples of such flows in the case of the flow past forward and backward steps and flow in cavities can be found in the Van Dyke album [7].

The author made calculations of the cases when a laminar vortex is pressed to the plane or is subject to shear. The Navier–Stokes equations were written in the stream function–vorticity variables. It was assumed that flow arises due to a rotating string placed in the surrounding medium. The presence of the string was modeled by a source term in the equation of vorticity transfer. The calculations show that in this case additional vortices usually appear in the flow. Fig. 1 presents the results of calculation of the stream function in a laminar vortical flow near the wall. For convenience of formulation of the boundary conditions it was assumed that the vortex is a long flat tube. It is natural that in the absence of walls the solution represented a set of concentric circles. As is seen from Fig. 1 the presence of walls not only deforms these circles but also stipulates the appearance of two additional vortices near the upper and lower walls of the tube. The asymmetry of the solution is caused by an asymmetric position of the string.

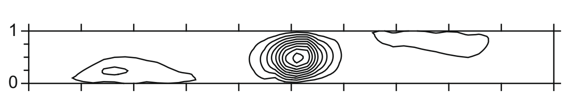

Fig. 1. Calculation of a laminar vortex flow in a cross section of the flat tube. String coordinates: $X = 5$, $Y = 0.47$.

It is evident that additional vortices exist at the expense of the energy of the same source that produces the initial vortex. Then we can state that due to the presence of walls and/or shear in the flow, the structure of the vortex, which appears first, turns to be such that it cannot take all the transferred energy and, as a result, a system of secondary vortices arises in the flow. But secondary vortices are also in contact with the wall and/or shear. Thus, as a result of this contact they also take not all the energy falling to their lot and tertiary, quaternary, and so on vortices can appear in the flow. Theoretically this process can continue infinitely.

It should be noted that results of the presented calculation can be interpreted in a wider sense. Any two-dimensional flow is a section of a three-dimensional flow. Two-dimensional equations of transfer are constructed on the basis of the hypothesis of the smallness of variables and/or derivatives in the direction perpendicular to the section plane. Then, it follows from the results given in Fig. 1 that the contact of an infinite straight vortex tube with the wall and/or shear generates additional vortex tubes in the flow. If the tube is bent, then, by easy parameterization of transport equations, we can pass to a straight tube, whence it follows that the effect of origination of additional vortices must exist on bent tubes as well. Dividing tubes into parts we obtain that the result is extended also to bent tubes of finite length.

At present, it is assumed to be established that turbulence, in its essence, is a set of vortex flows. Then, it follows from the calculations presented that the contact of a turbulent vortex, i.e., vortex tube, with a wall and/or the shear region will also generate additional vortices, or vortex tubes, in the flow.

4. The technique of turbulence model construction which allows to take into account the described effect

We consider the following model situation.

There is a single vortex in a homogeneous solid medium. Allowing for the fact that the flow energy is equal to the energy of the only vortex, the equation of the flow energy balance E can be written as follows:

$$dE/dt = P - \varepsilon.$$

Here, P is the velocity of energy supply to the vortex and ε is the rate of energy loss due to viscous friction against the medium.

Let now the vortex be pressed to the surface and/or be subject to the shear effect. In this case, the energy losses of the initial vortex increase sharply with at least one additional vortex appearing in the flow. Neglecting the energy of interaction of vortices we have that now the energy of flow is constituted of the energies of two vortices, i.e.,

$$E = E_0 + E_1,$$

where E_0 is the energy of the initial vortex and E_1 is the energy of the additional vortex. Then the equation of balance of the total energy of flow can be rewritten as follows:

$$d(E_0 + E_1)/dt = P - \varepsilon_0 - \varepsilon_1.$$

Here, ε_0 are the losses due to friction of the initial vortex with the medium and ε_1 are additional energy losses due to the contact of the initial vortex with the surface and/or the shear region.

It is obvious that some portion of the supplied energy P must be spent to the effects arising as a result of the contact with the wall and/or the shear region. Then the balance equation can be rewritten as

$$dE_0/dt = P - \varepsilon_0 + P_1 - \varepsilon_1 - dE_1/dt, \quad P = P_0 + P_1.$$

Here, $P_1 = dE_1/dt + \varepsilon_1$ is a portion of energy spent to the effects arising as a result of the contact with the surface and/or the shear region.

Now the equation of balance of the initial vortex energy can be written as

$$dE_0/dt = P_0 - \varepsilon_0.$$

We denote $P_0/P = f_0$, where f_0 is some function of the angular velocity of the vortex, the distance from the axis of rotation to the surface, etc. Hence, $P_1 = P - P_0 = (1 - f_0)P$. As a result, the equation of balance of the total energy disintegrates to two equations

$$\frac{dE_0}{dt} = f_0 P - \varepsilon_0,$$
$$\frac{dE_1}{dt} = (1 - f_0)P - \varepsilon_1.$$

As is shown by the calculations of laminar flows, the vortex with the energy E_1 is also pressed to the wall and/or is subject to shear. Thus, all the above considerations can be applied to it. Developing the theory, we obtain that the balance of the total energy of flow is described by an infinite system of equations of the form

$$\begin{cases} \frac{dE_0}{dt} = f_0 P - \varepsilon_0, \\ \frac{dE_1}{dt} = f_{01}(1 - f_0)P - \varepsilon_1 = f_{01}P_1 - \varepsilon_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dE_i}{dt} = f_{0i}(1 - f_{0i-1})P_{i-1} - \varepsilon_i = f_{0i}P_i - \varepsilon_i. \end{cases}$$

Neglecting as previously the energy of interaction of vortices we obtain that the total energy of flow consists of the sum of energies of all vortices $E = \sum E_i$.

It is known that turbulence is a vortex flow with the main portion of energy being concentrated in vortices having the dimensions comparable to the shear layer thickness. It is evident that these vortices are subject to effect of shear and/or touch the wall. In this case, as a result of the contact with the wall and/or the effect of shear these vortices generate in the flow the second vortex system that generates the third system, and so on. As a result the vortex system that generates turbulence can be presented as a set of an infinite number of vortex systems. Since just the presence of vortex systems in the flow creates fluctuations of instantaneous velocity, the resultant fluctuating component of velocity originates as result of superposition of fluctuations created by each vortex system. Hence it follows that $u = \sum u^i$, where u^i is the fluctuation generated by the ith system. In the long run, the Reynolds presentation of the instantaneous velocity $U = \overline{U} + u$ is substituted by the presentation $U = \overline{U} + \sum u^i$; in this case, it is assumed that fluctuations do not interact with each other, i.e., $\forall i \neq j \ \overline{u^i u^j} = 0$.

As a result transfer of total turbulent energy of the flow must be described by an infinite system of equations of the form

$$\begin{cases} \frac{Dk_0}{Dt} = \mathrm{Diff}(k_0) + f_0 P - \varepsilon_0, \\ \frac{Dk_1}{Dt} = \mathrm{Diff}(k_1) + f_{0-1}(1 - f_0)P_0 - \varepsilon_1 = \mathrm{Diff}(k_1) + f_{0-1}P_1 - \varepsilon_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{Dk_i}{Dt} = \mathrm{Diff}(k_i) + f_{0-i}P_i - \varepsilon_i. \end{cases} \tag{5}$$

Here, Diff is the operator of diffusion transfer (laminar and turbulent) $P \equiv -\overline{u_i u_j}(\partial U_i/\partial x_j), \quad P_i = (1 - f_{0-(i-1)})P_{i-1}$. It is natural that each equation of energy transfer must be supplemented by the corresponding equation of transfer of the dissipation rate.

We call the vortex system having the energy k_0 as primary vortices, the vortex system having the energy k_1 as secondary vortices, etc.

Now, taking into account system (5), the equations of the model for calculation of turbulence energy created by primary vortices can be written in the form

$$\begin{cases} \frac{Dk_0}{Dt} = \frac{\partial}{\partial x_k}\left(\nu + \frac{\nu_{t0}}{C_k}\right)\frac{\partial k_0}{\partial x_k} + f_0 P - \varepsilon_0 - E_{k0}, \\ \frac{D\varepsilon_0}{Dt} = \frac{\partial}{\partial x_k}\left(\nu + \frac{\nu_{t0}}{C_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_k} + \frac{\varepsilon_0}{k_0}(C_1 f_0 P - C_2\varepsilon_0) - E_{\varepsilon 0}. \end{cases} \tag{6}$$

$$P \equiv -\overline{u_i u_j}(\partial U_i/\partial x_j), \quad \nu_{t0} = C_\nu F_\nu k_0^2/\varepsilon_0. \tag{7}$$

The model is complemented by the relation for eddy viscosity of the Kolmogorov–Prandtl type (7). Here, ν_{t0} is the eddy viscosity created by the vortex system with energy k_0; E_k and E_ε are the corrections assigned to balance the diffusion on the wall.

It is assumed that all other vortex systems obey the same laws as primary vortices. Hence it follows that the equations of energy transfer of turbulence created by secondary, tertiary, etc. vortices can be written as

$$\begin{cases} \frac{Dk_i}{Dt} = \frac{\partial}{\partial x_k}\left(\nu + \frac{\nu_{t-i}}{C_{k-i}}\right)\frac{\partial k_i}{\partial x_k} + f_{0-i}P_i - \varepsilon_i - E_{ki}, \\ \frac{D\varepsilon_i}{Dt} = \frac{\partial}{\partial x_k}\left(\nu + \frac{\nu_{t-i}}{C_{\varepsilon-i}}\right)\frac{\partial \varepsilon_i}{\partial x_k} + \frac{\varepsilon_i}{k_i}(C_{1-i}f_{0-i}P_i - C_{2-i}\varepsilon_i) - E_{\varepsilon i}, \end{cases} \tag{8}$$
$$P_i = (1 - f_{0-(i-1)})P_{i-1}, \quad \nu_{t-i} = C_{\nu-i}F_{\nu-i}k_i^2/\varepsilon_i.$$

An assumption that all vortex systems obey the same laws imposes limitation on the correction terms and functions — they are calculated by absolutely the same regularities.

5. Calculation of deficient parameters of the new model of turbulence

To close model (6) it must first be supplemented by the expression for the function f_0. The graph of this function can be obtained based in the experimental data. In doing so it is enough to substitute into the model the experimentally obtained distributions of the turbulence energy and eddy viscosity and to consider system (6) as the system of equations relative to the function f_0 and dissipation.

But, in the judgment of the author, it is much more convenient to generate the values of the turbulence energy and eddy viscosity by calculating on the basis of the well-tested models of turbulence. It is obvious that since the energy of turbulence and eddy viscosity are the measurable parameters, all well-tested models must obtain them with rather high accuracy.

In the present paper, the models of Chien [8], Launder and Sharma [9], and Nagano and Tagawa [10] are used as data generators. The results of calculations of the function f_0 are given in Fig. 2. It is of interest to note that the graph almost exactly coincides with the velocity distribution in a laminar boundary layer. For comparison sake Fig. 2 gives the familiar Pohlhausen solution.

As a result of variance calculations the following approximation was obtained for the function f_0:

$$f_0 = \left(1 - \exp\left(-\frac{\mathrm{Re}_{y0}}{5.5}\right)\right)\left(1 - \exp\left(-2.4\frac{y}{L_{\varepsilon 0}}\right)\right), \tag{9}$$

where $\mathrm{Re}_{y0} = \sqrt{k_0}y/\nu, L_{\varepsilon 0} = k_0^{3/2}/\varepsilon_0$.

Test calculations have shown that the expressions

$$E_{k0} = (1 - f_0)\frac{\partial}{\partial x_k}\left(\nu + \frac{\nu_{t0}}{C_k}\right)\frac{\partial k_0}{\partial x_k}, \quad E_{\varepsilon 0} = (1 - f_0)\frac{\partial}{\partial x_k}\left(\nu + \frac{\nu_{t0}}{C_\varepsilon}\right)\frac{\partial \varepsilon_0}{\partial x_k}. \tag{10}$$

are good approximation for the wall corrections.

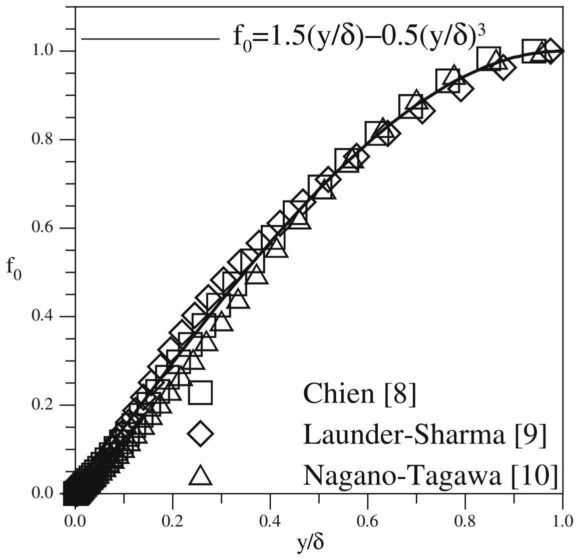

Fig. 2. Results of calculation of the function f_0.

After substitution of the wall corrections into the system of equations and cancelation we obtain the form of presentation of model equations

$$\begin{cases} \frac{Dk_0}{D\tau} = f_0 \frac{\partial}{\partial x_k}\left(\nu + \frac{\nu_{0i}}{C_k}\right)\frac{\partial k_0}{\partial x_k} + f_0 P - \varepsilon_0, \\ \frac{D\varepsilon_0}{D\tau} = f_0 \frac{\partial}{\partial x_k}\left(\nu + \frac{\nu_{t0}}{C_\varepsilon}\right)\frac{\partial \varepsilon_0}{\partial x_k} + \frac{\varepsilon_0}{k_i}(C_1 f_0 P - C_2\varepsilon_0). \end{cases} \tag{11}$$

By the results of test calculations for the function F_ν we obtained the following approximation:

$$F_\nu = \left(1 - \exp\left(-\frac{\mathrm{Re}_{y0}}{45}\right)\right)\left(1 - \exp\left(-2.4\frac{y}{L_{\varepsilon 0}}\right)\right). \tag{12}$$

The constants and the boundary conditions are:

$$C_\nu = 0.09, \quad C_\varepsilon = 1.3, \quad C_k = 1, \quad C_2 = 1.45, \quad C_1 = 0.9C_2;$$
$$y = 0 - U = k_0 = \varepsilon_0 = 0; \quad y \to \infty - U = U_e, k_0 = k_e, \varepsilon_0 = \varepsilon_e.$$

An analysis of the model functions reveals several unexpected coincidences with the known dependences. Any of these coincidences, taken separately, cannot evoke special interest, but simultaneous presence of them in the same model can likely serve an indirect proof of the correctness of the approach.

1. As has been already mentioned the graph of the function f_0 is in very good correspondence to the velocity distribution in the laminar boundary layer.
2. The obtained approximation of the function f_0 consists of two co-factors. The first of them acts in direct vicinity of the wall, the effect of the other propagates to the entire boundary region. The constant in the second co-factor can be presented as $2.4 = 1/\kappa$, where $\kappa = 0.416$ is the von Karman constant.
3. Approximation of the function F_ν also consists of the two co-factors. The second of them coincides with the function used in the approximation of f_0. The calculations show that the first co-factor virtually exactly corresponds to the well-known van Driest correction.

It is assumed that secondary, tertiary, etc. vortices obey the same laws as primary ones. By virtue of this the system of equations describing the transfer of energy produced by the ith vortex system must in all details, excepting maybe the constants, correspond to the system of equations describing the energy of primary vortices. In other words, to describe the energy of ith vortices we can use system (13)

$$\begin{cases} \frac{Dk_i}{Dt} = f_{0-i} \frac{\partial}{\partial x_k}\left(\nu + \frac{\nu_{ti}}{C_{k-i}}\right)\frac{\partial k_i}{\partial x_k} + f_{0-i}P_i - \varepsilon_i, \\ \frac{D\varepsilon_i}{Dt} = f_{0-i} \frac{\partial}{\partial x_k}\left(\nu + \frac{\nu_{ti}}{C_{\varepsilon-i}}\right)\frac{\partial \varepsilon_i}{\partial x_k} + \frac{\varepsilon_i}{k_i}(C_{1-i}f_{0-i}P_i - C_{2-i}\varepsilon_i), \end{cases} \tag{13}$$

$$P_i = (1 - f_{0-(i-1)})P_{i-1}, \quad \nu_{t-i} = C_{\nu-i}F_{\nu-i}\frac{k_i^2}{\varepsilon_i}, \quad \mathrm{Re}_{yi} = \frac{\sqrt{k_i}y}{\nu}, \quad L_{\varepsilon i} = \frac{k_i^{3/2}}{\varepsilon_i}$$
$$f_{0-i} = \left(1 - \exp\left(-\frac{\mathrm{Re}_{yi}}{5.5}\right)\right)\left(1 - \exp\left(-2.4\frac{y}{L_{\varepsilon i}}\right)\right),$$
$$F_{\nu-i} = \left(1 - \exp\left(-\frac{\mathrm{Re}_{yi}}{45}\right)\right)\left(1 - \exp\left(-2.4\frac{y}{L_{\varepsilon i}}\right)\right).$$

6. Testing the *k*–ε model: results of the calculation of forced turbulent flow in a boundary layer

Calculations were conducted for Re ranging from 5×10^5 to 1×10^7. In the calculations, the energies of the four first vortex systems of an infinite sequence were taken into account, i.e., calculations were made based on system (11) and three models of type (13). It was assumed that only primary vortices interact with the middle flow. This indicates that eddy viscosity $\nu_t = \nu_{t0}$ was used in the boundary-layer equations. In the models for calculation of primary vortices only one constant changed, viz, $C_{1\text{-}i} = 0.985C_2$.

Since approximation of the functions F_ν and f_0 was obtained on the basis of the well-tested models, we should expect that values of the averaged velocity and the friction coefficient must be obtained with good accuracy. Calculations confirm this hypothesis and thus are not presented here.

Figs. 3 and 4 give calculations of the energy of the four first vortex systems and turbulent friction produced by these systems. As is seen from Fig. 3, the total energy of primary and secondary vortices is virtually in accurate correspondence with the experiments of Klebanoff mentioned in [5]. The calculated turbulent friction created by primary vortices corresponds to the experimental data of Klebanoff, which is in agreement with the assumption that $\nu_{t0} \approx \nu_t$. The results of calculations of the tertiary and quaternary systems

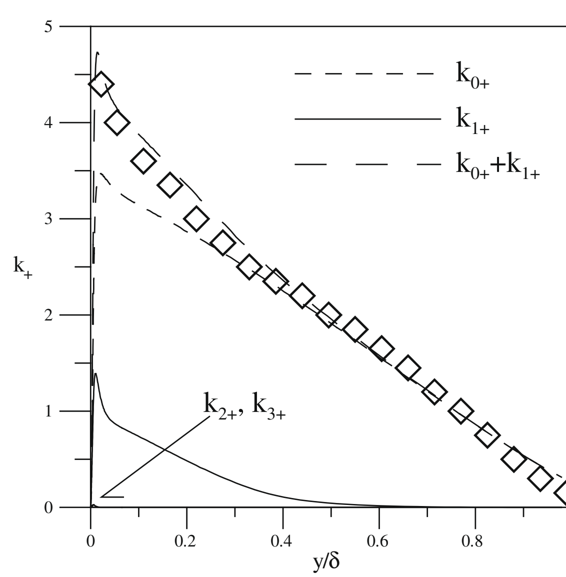

Fig. 3. Calculation of the energy of the first four vortical systems. Symbols – Klebanoff, see [5].

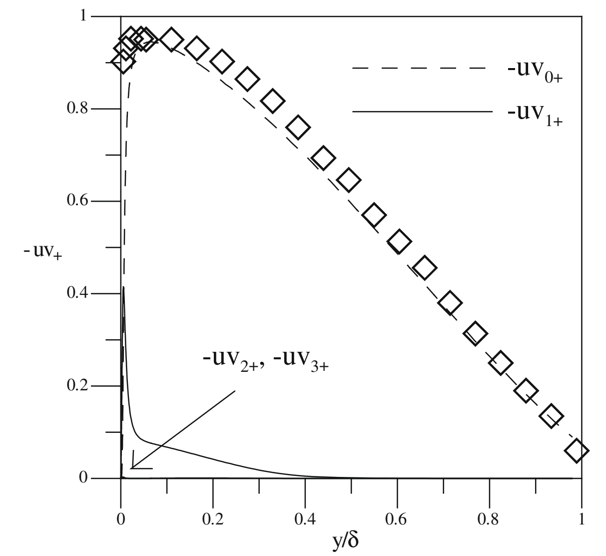

Fig. 4. Calculation of turbulent friction produced by the first four vortical systems. Symbols – Klebanoff, see [5].

were not displayed in the scale of graphs, i.e., they are negligibly small. Hence it follows that in calculation of real flows, if we are not interested in accurate values of the energy of turbulence, we can manage only with calculations of the primary vortices.

Fig. 5 presents the results of calculation of dissipative scales of the first two vortex systems. The dissipative scales of tertiary and quaternary vortices also were not displayed in the scale of the graph. We note that at $\mathrm{Re} = 5 \times 10^6$ the calculations show that maximum values of dissipative scales of the four vortex systems are $L_{\varepsilon 0\max} \approx \delta$, $L_{\varepsilon 1\max} \approx 0.075\delta$, $L_{\varepsilon 2\max} \approx 0.013\delta$, $L_{\varepsilon 3\max} \approx 10^{-5}\delta$. Allowing for the values of turbulence energy shown in Fig. 3 we obtain that each following vortex system exists mainly as perturbation against the background of the previous one. Hence it follows that neglecting their interaction does not introduce any noticeable distortions into calculation.

The calculations show that primary and secondary vortices greatly differ by the scales of time, i.e., the time of existence. This fact also confirms the possibility of neglecting the energy of interaction of vortex systems.

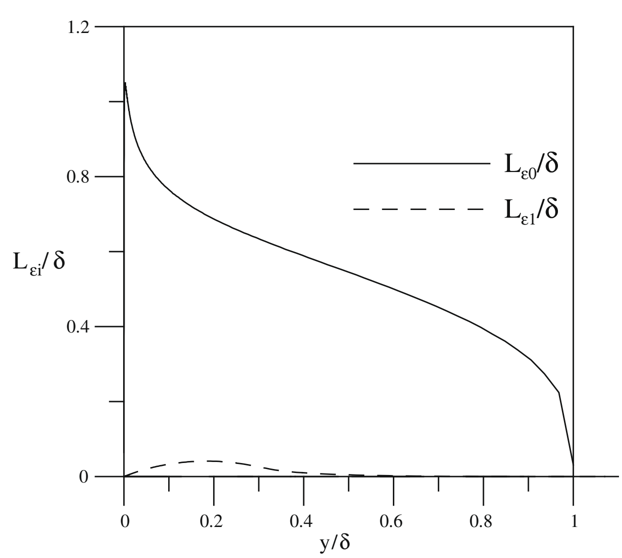

Fig. 5. Dissipative scales of the first two vortical systems.

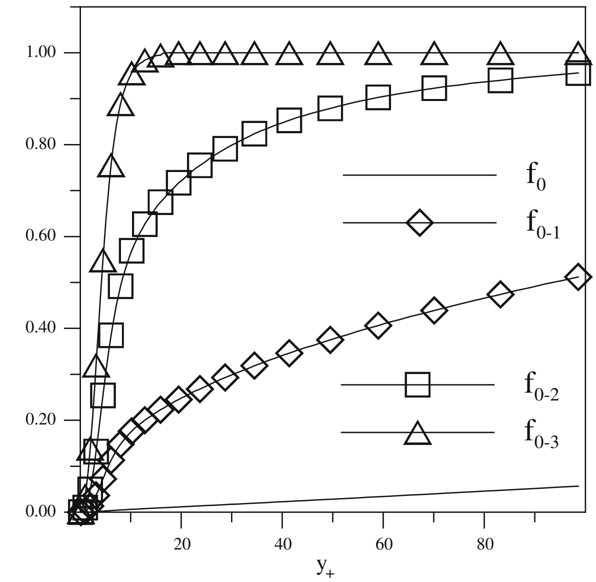

Fig. 6. Calculation of the function f_0 for four vortical systems.

Fig. 6 demonstrates the result of calculation of the function f_0 for three vortex systems. The function of the fourth vortex system on the mesh used in calculation was equal to unity. Hence it follows that at the fifth vortex system the sequence of vortex systems is virtually interrupted.

7. Correctness of the suggested model

Making use of the calculations performed we show that the suggested splitting of the instantaneous velocity is per se the model presentation of the exact Reynolds equation and does not introduce any new terms into it except for the traditionally used wall corrections.

First, we note the following fact. The equation of transfer of k in the system (6) after substitution into it of the expression for E_{k0} (10) can be rewritten as follows:

$$\frac{Dk_0}{D\tau} = \nu\frac{\partial^2 k_0}{\partial x_k \partial x_k} + f_0\frac{\partial}{\partial x_k}\frac{\nu_{t0}}{C_k}\frac{\partial k_0}{\partial x_k} + f_0 P - \varepsilon_0 - (1 - f_0)\nu\frac{\partial^2 k_0}{\partial x_k \partial x_k}.$$

This writing is equivalent to the interpretation of the system of equations as

$$\mathrm{Diff}_{\mathrm{turb}}(k_0) = f_0\frac{\partial}{\partial x_k}\frac{\nu_{t0}}{C_k}\frac{\partial k_0}{\partial x_k}, \quad E_{k0} = (1 - f_0)\nu\frac{\partial^2 k_0}{\partial x_k \partial x_k} \tag{14}$$

This interpretation does not change the form of system (11) but allows better understanding of the model properties. Now, we can write system (5), which describes transfer of the total energy of turbulence, as

$$\begin{cases} \frac{Dk_0}{D\tau} = \nu\frac{\partial^2 k_0}{\partial x_k \partial x_k} + f_0\frac{\partial}{\partial x_k}\frac{\nu_{t0}}{C}\frac{\partial k_0}{\partial x_k} + f_0 P_0 - \varepsilon_0 - E_{k0,} \\ \frac{Dk_1}{D\tau} = \nu\frac{\partial^2 k_1}{\partial x_k \partial x_k} + f_{0-1}\frac{\partial}{\partial x_k}\frac{\nu_{t1}}{C}\frac{\partial k_1}{\partial x_k} + f_{0-1}P_1 - \varepsilon_1 - E_{k1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{Dk_i}{D\tau} = \nu\frac{\partial^2 k_i}{\partial x_k \partial x_k} + f_{0-i}\frac{\partial}{\partial x_k}\frac{\nu_{ti}}{C}\frac{\partial k_i}{\partial x_k} + f_{0-i}P_i - \varepsilon_i - E_{ki}. \end{cases} \tag{15}$$

Here, the terms E_{ki} are the wall corrections of the form of (14), $P_0 \equiv \nu_{t0}\left(\frac{\partial U_k}{\partial x_j} + \frac{\partial U_j}{\partial x_k}\right)\frac{\partial U_k}{\partial x_j}$ $\quad P_i = (1 - f_{0-(i-1)})P_{i-1}$.

The equation describing transfer of the total energy of turbulence is obtained by summation of equations of (15)

$$\sum \frac{Dk_i}{D\tau} = \sum \nu \frac{\partial^2 k_i}{\partial x_k \partial x_k} + \sum f_{0-i} \frac{\partial}{\partial x_k} \frac{\nu_{t-i}}{C_k} \frac{\partial k_i}{\partial x_k} + \sum f_{0-i} P_i - \sum \varepsilon_i - \sum E_i. \qquad (16)$$

As has been already mentioned the calculations show that the functions $f_{0\text{-}i}$ used in Eq. (15) are equal to unity over the entire thickness of the layer, except for a small region near the wall, with this region decreasing very quickly as i increases, i.e., $\lim_{i\to\infty} f_{0-i} \equiv 1$. Then it is not difficult to show that

$$\lim_{i\to\infty} \sum_{j=0}^{i} f_{0-i} P_i = P_0 \lim_{i\to\infty} \sum_{j=0}^{i} \left(f_{0-j} \prod_{k=j+1}^{i} (1 - f_{0-k}) \right) = P_0 = P. \qquad (17)$$

Substituting (17) in (16) and using obvious relations

$$\sum \frac{Dk_i}{D\tau} = \frac{D\sum k_i}{D\tau} = \frac{Dk}{D\tau}, \quad \sum \nu \frac{\partial^2 k_i}{\partial x_k \partial x_k} = \nu \frac{\partial^2 \sum k_i}{\partial x_k \partial x_k} = \nu \frac{\partial^2 k}{\partial x_k \partial x_k},$$

we obtain the equation of transfer of the total energy of turbulence

$$\left\{ \frac{Dk}{D\tau} = \nu \frac{\partial^2 k}{\partial x_k \partial x_k} + \sum f_{0-i} \frac{\partial}{\partial x_k} \frac{\nu_{t-i}}{C_k} \frac{\partial k_i}{\partial x_k} + P - \sum \varepsilon_i - \sum E_{ki} \right\}$$
$$= \left\{ \frac{Dk}{D\tau} = \nu \frac{\partial^2 k}{\partial x_k \partial x_k} + \mathrm{Diff}_{\mathrm{turb}}(k) + P - \varepsilon - E \right\}.$$

Here, we used the notation $\mathrm{Diff}_{\mathrm{turb}}(k) \equiv \sum f_{0-i} \frac{\partial}{\partial x_k} \frac{\nu_{t-i}}{C_k} \frac{\partial k_i}{\partial x_k}$, $\quad \varepsilon \equiv \sum \varepsilon_i$, $E \equiv \sum E_{ki}$.

Thus, this model does not introduce additional terms and does not loses the terms in the Reynolds equations, with the exact terms of the model in the limit being correspondent to the terms of these equations. This indicates that the model is a correct, from the viewpoint of modern models (see, e.g., [11]), presentation of the Reynolds equations.

8. Calculation of the boundary-layer flow with a large positive pressure gradient

Model (11) was verified by calculations of the boundary-layer flow with a large positive pressure gradient (experiments of Samuel and Joubert [12]). Figs. 7 and 8 present the results of calculations of the friction coefficient and the profiles $\overline{u_0 v_{0+}}$ in three cross sections of the channel. Correlation $\overline{u_0 v_0}$ was calculated according to the Boussinesq hypothesis. The agreement with the experimental data is satisfactory (X is the distance from the beginning of the boundary layer).

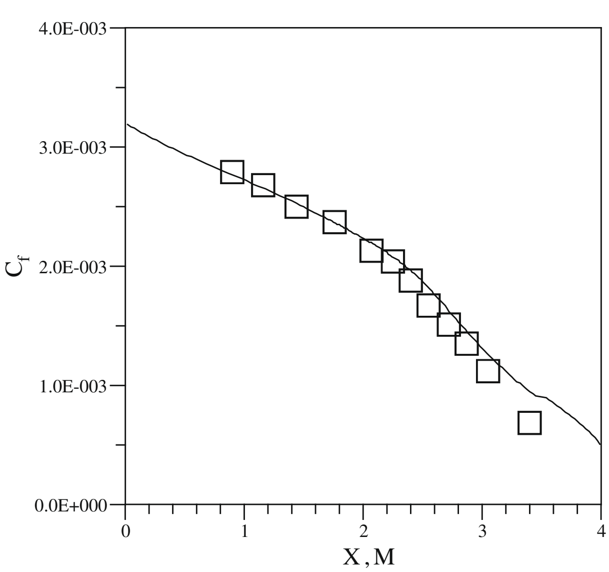

Fig. 7. Friction coefficient in the flow with laminarization and re-laminarization.

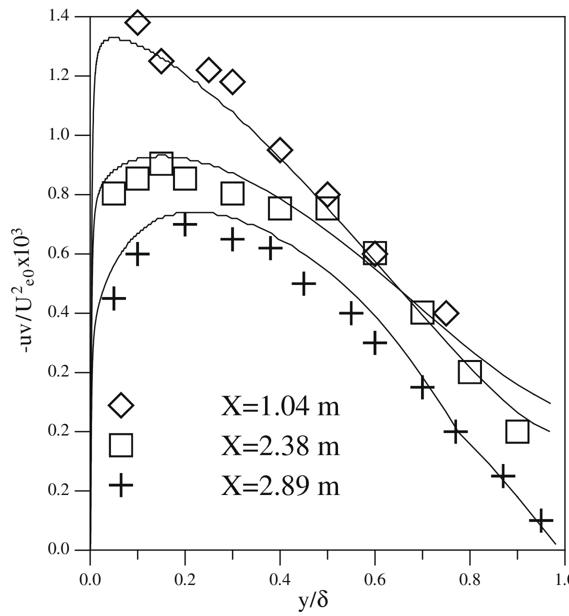

Fig. 8. Turbulent friction in the flow with laminarization and re-laminarization.

We note that most of the calculations of this flow known to the author(see the works of Rodi and Scheurer [13] and Nagano and Tagawa [10]), which have a similar agreement between the calculations of friction and the experimental data, are per se adjustment, since the main method underlying these calculations is as follows. Generation of k dependent on $\partial U/\partial x$, viz $\partial U/\partial x(\overline{u^2} - \overline{v^2})$, is introduced into the model, but in the equations of k and ε transfer it is allowed for by different coefficients. In the present calculations such methods were not required.

9. Modeling of the bypass laminar-to-turbulent transition

The problem of bypass laminar-to-turbulent transition in a boundary layer on a flat plate or transition at elevated turbulence of the outer flow plays an important role in designing the turbine units. This problem cannot be accepted solved. Thus, for example, in Ref. [14], the existing k–ε models were systematically tested for suitability to calculate bypass transition. The results obtained show that none of the tested models can correctly calculate either the beginning of transition or the length of the transition region.

So, in a uniform turbulent flow a flat plate is positioned parallel to the main flow. Distributions of all fluctuating and averaged parameters of the flow in the outer flow and in front of the plate are known. At some distance from the beginning of the plate, due to the effect of outer turbulence, the characteristics of the boundary layer, which develops on the plate, begin to differ from the characteristics of the laminar boundary layer, i.e., transition starts. With time the layer on the plate becomes turbulent. It is necessary to calculate the process of turbulence development in the boundary layer. The initial conditions must be assigned in the form of rectangular profiles of all parameters directly at the initial point of the plate, i.e., in a single physically justified form.

To solve this problem the model required some modification. As is known, generation of the correlation $\overline{uv}$ is described by the expression $-\overline{v^2}\partial U/\partial y$. Therefore, the correlation $\overline{uv}$ must be proportional to $-\overline{v^2}\partial U/\partial y$ whence it follows that ν_t is proportional to $\overline{v^2}$. In calculation of eddy viscosity by the expression $\nu_t \equiv \nu_{t0} = C_\nu F_{\nu-0} k_0^2/\varepsilon_0$ the ratio between $\overline{v_0^2}$ and k_0 is allowed for by the function $F_{\nu\text{-}0}$. It is known from the experiments that in the case of the transition layer this ration strongly differs from the ratio in the case of the developed turbulence. For this reason, an additional factor ϕ_v, which allows for this difference in the laminar mode and which is equal to unity in the developed turbulent mode, was introduced into the function F_v

$$\varphi_v = 1 - C_0 \exp(-2.5\sqrt{v_t}) \Big/ \exp\left(0.25\left(\frac{C_f}{C_{f\_lam}} - 1\right)^2\right)$$

Here, $C_{f\_lam} = 0.664/\sqrt{\mathrm{Re}}$ is the friction coefficient in the laminar mode of flow, $C_0 = 1^{-1} \times 10^{-5}$.

In the calculations, an exact solution of the problem on the final stage of free turbulence decay was used as boundary conditions.

The initial conditions were specified at the point correspondent to Re = 1. Rectangular profiles of all parameters were used as the initial conditions. The results of calculations were compared with the experimental data known in the literature as T3A and T3B [15].

Figs. 9 and 10 give the results of calculations of the friction coefficient and correlation $\overline{u_0 v_0}$. The correlation $\overline{u_0 v_0}$ was calculated according to the Boussinesq hypothesis. For comparison, these figures present the experimental data of Roach and Brierley [15]. Fig. 10 also shows the results of the LES computation by Yang and Voke [16]. We note that the results of calculations of the form parameter H, mean velocity, and energy of turbulence are also in good agreement with the experimental data.

It should be mentioned that the model allows not only calculation of the friction coefficient, profiles of velocity and fluctuating quantities, etc. Thorough analysis of the calculations makes it possible to reveal such regularities of the bypass transition, which earlier were not even attempted. So, for example, a minimum point of loss of boundary layer stability, which was found by the results of calculations, correspond to $\mathrm{Re}^{**} \approx 162$. At the same time, a minimum point of loss of stability found from the solution of the Orr–Sommerfeld equations is $\mathrm{Re}^{**} = 164$ [17]. The coincidence is virtually exact.

10. Analogies with the coherent structures, which arise in the calculations by the suggested models

An analysis of the calculations shows that secondary vortices are in good agreement with some theoretical and experimental data on coherent structures. For example, Zhang and Lilley [18] suggested the model for calculation of the development of disturbances in the boundary layer. In this mode, periodic disturbances were imposed on a turbulent flow. As a result, secondary disturbances in the form of slowly rotating formations appeared in the flow. The authors identified these secondary disturbances as coherent structures. Fig. 11 presents comparison of the calculations of the disturbance of the middle flow obtained by Zhang and Lilley [18] with the energy of secondary turbulence. The agreement is very good.

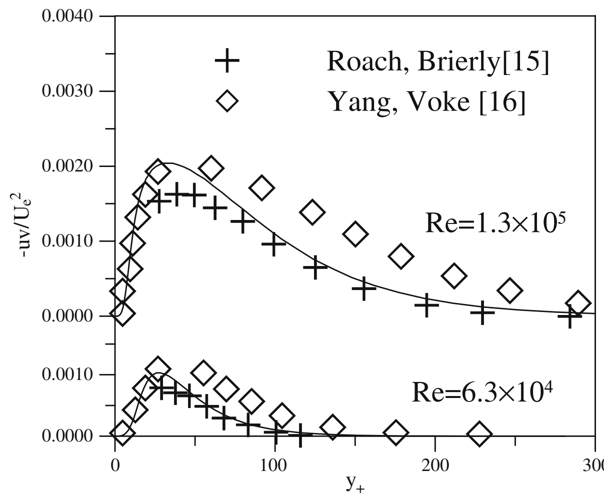

Fig. 10. Bypass transition. Turbulent friction.

Thus, we can state that this calculation reveals some correspondence with the data on coherent structures. It should be noted that this correspondence is not single.

In his review [19], Cantwell indicates that in the wall region coherent vortices of two types play an important role in a turbulent boundary layer. First, the wall is covered by a system of longitudinal vortices with a typical diameter $\lambda_y \approx (20 \div 30)\nu/u$. Fig. 12 shows the distribution of the energy of tertiary turbulence. It is well seen from the figure that distribution of the tertiary turbulence is in good correspondence with the dimensions of these vortices. Second, vortices with a typical diameter $l_y \approx (20 \div 40)\nu/u^*$ lie above them. If we assume that these vortices touch the wall,

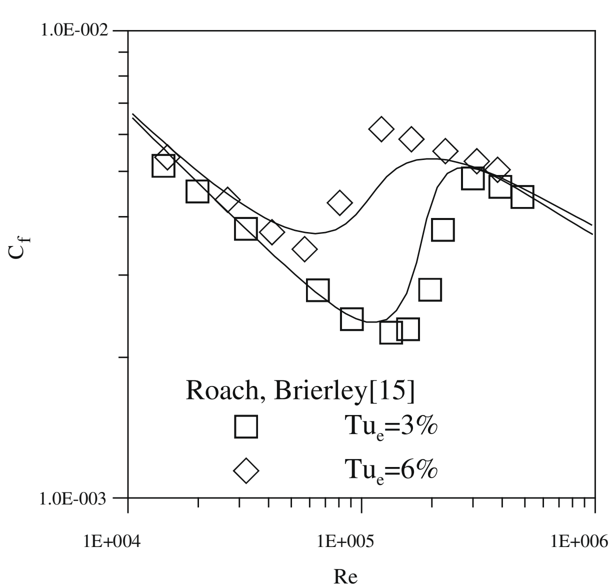

Fig. 9. Bypass transition in the boundary layer. Friction coefficient.

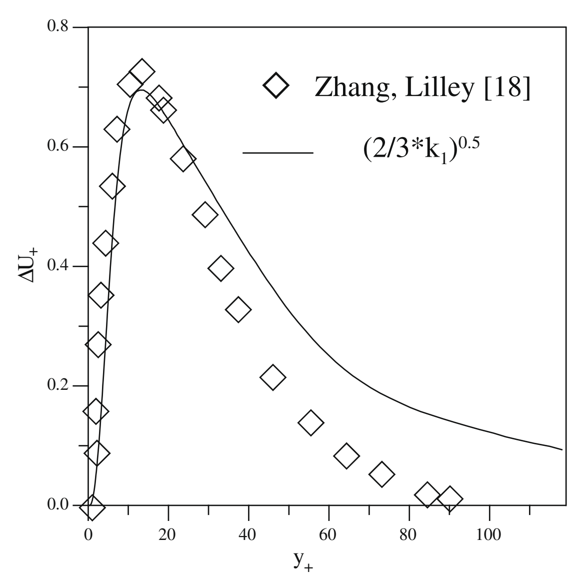

Fig. 11. Energy of secondary turbulence and coherent structure according to Zhang and Lilley [18].

then they must lie in the region $l_y \approx (50 \div 70)\nu/u^*$ (the dimensions are obtained by summing the typical sizes of two vortex systems). Fig. 11 meets these requirements quite well.

11. Modeling of the cascade process

It is often ascertained that in the k–ε models ε indicates transfer of energy into the cascade process. But due to the presence of corrections in the model equations, ε, generally speaking, denotes the imbalance of the equation of energy transfer. The fact that this imbalance corresponds to energy transfer into the cascade process must be proved. The only proof may be the possibility of calculation based on the model of cascade energy transfer. Calculation may be performed in the following way. At the first step, the system of the k–ε type is solved. At the second step, the same system is solved with the dissipation, which is obtained at the first step, being used instead of the generation term. It is evident that the calculation step simulates the step of the cascade process. The number of steps is not limited.

The correspondence of the results of calculation to the cascade process can be verified in the following way. First, it is known that transition of turbulent energy to heat occurs in vortices that have the dimensions commensurable with the Kolmogorov scales, i.e., the energy transferred to the cascade does not change. Hence it follows that if the dissipative term reflects the transfer of energy into the cascade process, then in such calculation it must not change.

Second, if we assume that the wave number is proportional to $1/L_\varepsilon$, which follows from the dimensional reasons, then the dependence of k_i/k on $1/L_i$, where i is the number of the step of the mentioned calculation, must not, as a minimum, be in contradiction with the known regularities of the cascade transfer.

In the calculations by this model, the requirement $\varepsilon = \text{const}$ is satisfied almost exactly. At the same time, the turbulence energy, dissipative scales, and scales of time decrease noticeably from step to step.

The dependence of k_i/k on $1/L_i$ shown in Fig. 13 is in good agreement with the "$-5/3$" law.

In the judgment of the author, these results prove that in the present model ε reproduces just the rate of energy transfer to the cascade process. In other words, ε has a specific physical meaning, which cannot be said of the traditional models. In particular,

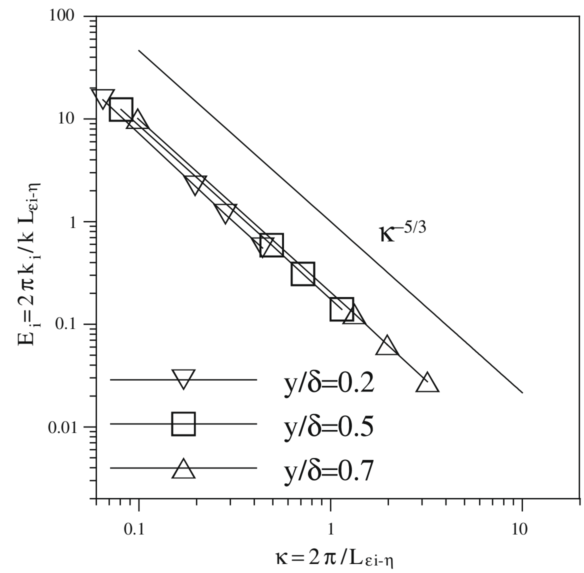

Fig. 13. Discrete spectrum of turbulent energy in three cross sections of the boundary layer.

calculation of the cascade process by the Nagano–Tagawa model [10] showed that in this case, in development of the cascade process the vortices are shifted toward higher, rather that lower, wavenumbers.

A very interesting picture is given by the calculation of the cascade process occurring with the secondary turbulence. The calculations show that in this case the energy virtually does not change from step to step. At the same time, calculations of the time scales k/ε of the primary and secondary turbulence show that the time scale of the secondary turbulence is several times lower than the time scale of the primary turbulence. Hence it follows that in contrast of the primary vortex the secondary vortex decomposes in fact very quickly. At the same time, a new vortex that appears on its place virtually coincides with the initial one which produces an effect of a very large life cycle of the secondary vortex. We can assume that just this fact makes coherent structures observable.

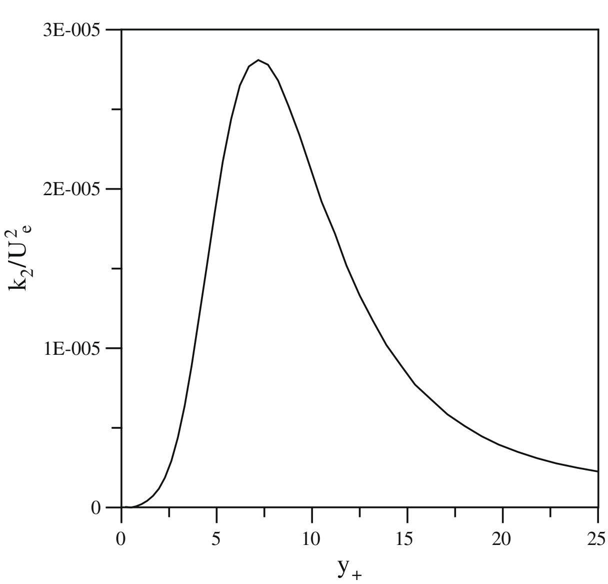

Fig. 12. Distrubution of the energy of tertiary turbulence.

12. The $\overline{t^2}$–ε_t model for calculation of boundary-layer flows

We consider a single vortex in a nonuniformly heated medium. It is known that as a result of entrainment by the vortex of more heated liquid layers and transfer of them to less heated ones and vice versa temperature fluctuations are created in the liquid. Hence it follows that in an isothermal turbulent flow there should exist temperature fluctuations generated by primary, secondary, etc. vortices. It is clear that each of these temperature fluctuations must be described by a separate system of equations.

It follows from the fact that temperature fluctuations are produced by velocity fluctuations that the structure of the equations describing the transfer of temperature fluctuations must correspond to the structure of the equations describing the transfer of velocity fluctuations. At the same time, when $\Pr > 1$ the model must be complemented with the account for the following physical effect. It is known that temperature fluctuations are produced in liquid as a result of entrainment by turbulent vortices of more heated liquid layers and transfer of them to less heated and vice versa. But due to the fact that $\Pr > 1$, vortices, as kinematic formations, must decompose quicker than temperature fluctuations are eliminated by heat conduction. As a result, temperature inhomogeneity will exist at the place of each vortex. It seems that this phenomenon cannot be reproduced accurately within the turbulence model. From the viewpoint of turbulence modeling, this phenomenon can be treated as follows: turbulence takes more thermal energy from the averaged flow than can be dissipated or, in other words, not all the energy described by generation terms is used to maintain fluctuations. This problem can be solved by introduction of additional dissipation $\varepsilon_{t\text{-add}}$ into the model. This dissipation is calculated by the formula

$$\varepsilon_{t\text{-add}} = \max\left(0.5\overline{t^2}\frac{\varepsilon}{k} - \varepsilon_t, 0\right). \quad (18)$$

It follows from (18) that $\varepsilon_{t\text{-add}}$ is calculated such that the fulfillment of the inequality $0.5\overline{t^2}/(\varepsilon_t + \varepsilon_{t-\text{add}}) \le k/\varepsilon$ is guaranteed, i.e., the time scale of temperature fluctuations did not exceed the time scale of kinematic fluctuations.

Allowing for the structure of the model of transfer of energy of primary vortices (11) and the expression for additional dissipation (18) we obtain the model for calculation of transfer of temperature fluctuations created by primary vortices

$$\begin{aligned}
&\frac{D\overline{t_0^2}}{D\tau} = f_{0-t}\frac{\partial}{\partial x_k}\left(\alpha + \frac{\alpha_{t0}}{\sigma_t}\right)\frac{\partial\overline{t_0^2}}{\partial x_k} + 2f_{0-t}P_t - 2(\varepsilon_{t0} + \varepsilon_{t0\text{-add}}),\\
&\frac{D\varepsilon_{t0}}{D\tau} = f_{0-t}\frac{\partial}{\partial x_k}\left(\alpha + \frac{\alpha_{t0}}{\sigma_{\varepsilon t}}\right)\frac{\partial\varepsilon_{t0}}{\partial x_k} + \frac{\varepsilon_{t0} + \varepsilon_{t0\text{-add}}}{0.5\overline{t_0^2}}(C_{1t}f_{0-t}P_t - C_{2t}\varepsilon_{t0}),\\
&P_t = -\overline{u_{i0}t_0}\frac{\partial\overline{T}}{\partial x_i}.
\end{aligned} \quad (19)$$

Here $f_{0\text{-t}}$ is the function describing the effect of the wall on temperature fluctuations.

The system of Eq. (19) is closed by the relations for turbulent thermal diffusivity α_{t0} and the correlation $\overline{u_{i0}t_0}$

$$\begin{aligned}
&\alpha_{t0} = C_\lambda F_\lambda\left(k_0, \varepsilon_0, \varepsilon_{t0}, \overline{t_0^2}, \ldots\right)k_0\frac{0.5\overline{t_0^2}}{\varepsilon_{t0} + \varepsilon_{t0\text{-add}}},\\
&\overline{u_{i0}t_0} = -\alpha_{t0}\frac{\partial\overline{T}}{\partial x_i}.
\end{aligned} \quad (20)$$

The function $f_{0\text{-t}}$ was calculated on the basis of two models of Nagano and coworkers [20,21] by the same method as the function f_0. At Pr numbers of the order of unity the results of calculations showed practically full coincidence of the functions f_0 and $f_{0\text{-t}}$. This coincidence reflects the known hypothesis (see, e.g., [22]) according to which the field of velocity fluctuations controls the field of fluctuations of a scalar quantity whence it follows that the wall affects the transfer of temperature fluctuations not directly but via interactions with the turbulent medium, which is expressed by the equality $f_{0\text{-t}} = f_0$.

By the results of calculations for $f_{0\text{-t}}$ we choose the following approximation

$$\begin{aligned}
&f_{0-t} = \left(1 - \exp\left(-R_0\frac{\mathrm{Re}_{y0}}{5.5}\right)\right)\left(1 - \exp\left(-2.4\frac{y}{L_{\varepsilon 0}}\right)\right),\\
&\mathrm{Re}_{y0} = \frac{\sqrt{k_0}y}{\nu}, \quad L_{\varepsilon 0} = \frac{k_0^{3/2}}{\varepsilon_0}.
\end{aligned} \quad (21)$$

Here, $R_0 = \left(\frac{k_0}{\varepsilon_0}\Big/\frac{0.5\overline{t_0^2}}{\varepsilon_{t0}+\varepsilon_{t0\text{-add}}}\right)$ is the ratio of time scales.

For F_λ [see expression (20)] we obtain the approximation

$$F_\lambda = \left(1 - \exp\left(-R_0\frac{\mathrm{Re}_{y0}}{45}\right)\right)\left(1 - \exp\left(-2.4\frac{y}{L_{\varepsilon 0}}\right)\right). \quad (22)$$

The parameter R_0, as in (22), denotes the ratio of time scales. The constants and the boundary conditions are

$$C_{2t} = 1.45, \quad C_{1t} = 0.9C_{2t}, \quad \sigma_t = 1, \quad \sigma_{\varepsilon t} = 1.3.$$

$$y = 0 - \overline{t_0^2} = \varepsilon_{t0} = 0, \quad T = T_\mathrm{w} = \mathrm{const}\left(-\lambda\frac{\partial T}{\partial y} = Q_\mathrm{w} = const\right).$$

$$y \to \infty - T = T_\mathrm{e} = \mathrm{const}, \quad \overline{t_0^2} = \overline{t_{0\mathrm{e}}^2}, \quad \varepsilon_{t0e} = \varepsilon_{te}.$$

Testing of the model shows that the constant C_λ slightly depends on Pr. For Pr > 1, good results are obtained at $C_\lambda = 0.09$; for Pr = 0.72 the value $C_\lambda = 0.1$ was used. Due to the absence of reliable experimental data it is impossible to construct any functional relation. We emphasize that calculations for Pr << 1 were not performed in the present work.

Fig. 14 shows the results of calculations of forced-convection heat transfer in a turbulent boundary layer. The results of calculations are compared with the approximation of Zukauskas [23]

$$0.5C_f/St = 0.93 + 12.5\sqrt{0.5C_f}(\mathrm{Pr}^{2/3} - 1)$$

The agreement is very good.

13. Experimental confirmation of the theoretical grounds of the approach

We briefly recall the main hypotheses underlying the approach. It is assumed that the main role in the turbulent boundary layer is played by vortices with the sizes commensurable with the boundary-layer thickness. These vortices are called primary vortices. Thanks to the contact with the wall and/or shear, these vortices cannot accept all energy transferred to them from the middle flow. As a result, a system of secondary vortices appears in the flow. These vortices are also in contact with the wall and/or shear. In the end, tertiary, quaternary, etc. vortices arise in the flow.

Repik and Sosedko [24] thoroughly analyzed the existing results of the investigation of the processes of viscous sublayer renewal in the wall region of the flow. Experimental studies show that the main role in the boundary layer is played by large-scale quasi-ordered structures commensurable with the boundary-layer thickness. These structures move along the flow at the velocity approximately two times smaller than the velocity outside the boundary layer. As the structures move the velocity at the fixed point of the space within one structure gradually decreases in time. After passing the boundary that closes this structure, intense high-frequency fluctuations of velocity, temperature, pressure, etc. are observed

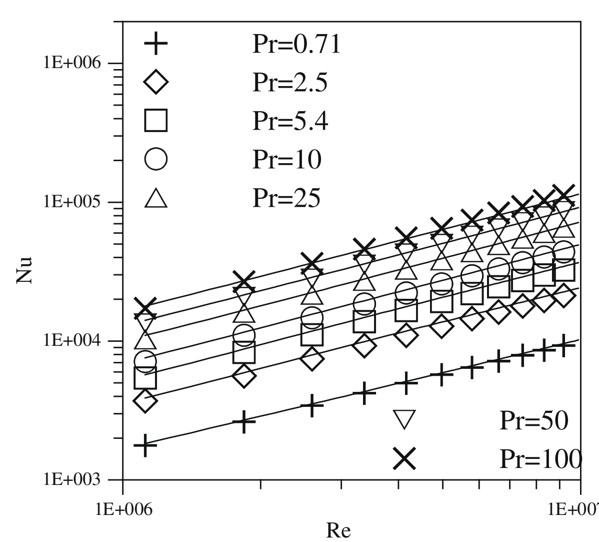

Fig. 14. Heat transfer in forced convection in the boundary layer. Symbols – Zukauskas [23].

and there occur jet ejections of decelerated liquid from the wall and invasion of accelerated to the wall region. As a result of complex perturbations, the so-called coherent structures are formed in the liquid from jet ejections. All this is a single cycle of sublayer renewal.

The experiments show that jet ejections of liquid are accompanied by the appearance of the inflection point on the profile of the average velocity. The reason of its appearance is in the interaction of the accelerated and decelerated regions of two large-scale structures. In turn, the appearance of the inflection point provides the appearance of a new transversely oriented vortex. Hence it follows that a transverse vortex originates and thus exists at the expense of the energy of averaged flow.

As stated by Repik and Sosedko [24], the described processes can be studied either visually or selecting a corresponding, rather small, time of averaging for calculation of each elementary process. In constructing the models of turbulence a very large time of averaging, which theoretically tends to infinity, is used. As a result of such averaging, the following picture arises from the described process. A set of large-scale transverse vortices is present in the layer. These vortices exist due to the energy of the average flow. As a result of interaction of these vortices with each other and with the wall a set of medium-scale vortices, which are identified as coherent structures, appears in the flow.

At the same time, as has been mentioned already, origination of medium-scale vortices is accompanied by retardation of large-scale vortices. Thus, we can state that medium-scale vortices take the energy from the large-scale vortices or a portion of energy that is intended to create and maintain large-scale vortices is spent for creation and maintenance of medium-scale vortices.

In the judgment of the author there exists an obvious qualitative coincidence between the described mechanism and theoretical prerequisites of the suggested approach.

14. Conclusions

In the paper, an approach to construction of the models of turbulence, which allows modeling of the fluctuating components as a sum of an infinite number of random quantities, is suggested. As a result, total kinetic energy of turbulence is presented in the form of the sum of an infinite number of components and its transfer is described by an infinite number of similar models of the k–ε type. But an infinite number of components, systems of equations, etc. is present in the model only theoretically. The calculations show that in calculation of real flows it is suffice to do only with the first system of equations. According to the assumptions of the suggested theory, all subsequent systems only verify the value of the turbulence energy and do not interact with the middle flow. The calculations show that in real flows, tertiary, quaternary, etc. systems can be neglected.

In the end, the range of problems easy for modeling has been expanded considerably. In the list of the solvable problems there appear such problems as calculation of bypass transition, cascade transfer of energy, calculation of coherent structures, etc., i.e., the problems that have not been posed earlier.

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